Applications of Graphic Statics to Analysis and Design of Reinforced Concrete: Stress Fields and Yield Lines

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Abstract

Graphic statics can be effectively used for the analysis and design of structures in static equilibrium and for solving engineering problems related to specific materials, most notably, reinforced concrete. In this regard, methods for the construction of stress fields and yield lines, based on the theory of plasticity, are particularly valuable; however, their automation and generalisation to the third dimension are still open research topics. This paper introduces a unified method based on graphic statics for the generation of discrete stress fields of 2D and 3D reinforced concrete structures and the definition of admissible yield line patterns in concrete slabs. The presented approach relies on the construction of reciprocal stress functions, and the link between form and force diagrams through Minkowski sums.

1 Introduction

1.1 Graphic statics

Graphic statics is a 19th-century geometry-based method for the analysis and design of structures in static equilibrium grounded on the notion of reciprocity between form and force diagrams. Although largely abandoned in the early 20th century, due to the development of analytical mechanics, in the 21st century, graphic statics has experienced a resurgence owing to the advanced computational and visualisation capabilities of contemporary digital tools. As a result, several new methodologies and design frameworks based on graphic statics have been developed over the last few years. However, most of these applications are based on iterative optimisation algorithms that take advantage solely of reciprocal form and force diagrams, while neglecting the importance of their underlying higher dimensional reciprocal stress functions. Only recently, the fundamental role of the stress functions in relation to graphic statics has been brought forward. In this regard, it has been shown that 2D trusses in static equilibrium are projections of 3D polyhedral Airy stress functions (Mitchell et al. [13]), in which the change of slope between adjacent faces defines the axial force in the corresponding structural member of the projected 2D truss. Equivalently, polyhedral (plane-faced) 3D trusses are projections of 4D stress functions (McRobie [12]).

1.2 Strut-and-tie models and stress fields

Strut-and-tie models and stress field models can be efficiently used for the analysis and design of reinforced concrete structures in static equilibrium since they provide an effective way to represent the flow of internal forces and stresses. These two models, which are both based on the lower bound theorem of the theory of plasticity, are complementary and closely related to each other. In particular, a strut-and-tie model can be regarded as a simplified version of an underlying stress field and obtained from the latter after defining the stress resultants (Muttoni et al. [16]).

On the one hand, strut-and-tie models were initially introduced by Schlaich et al. [18] as a generalisation of the earlier truss models, to describe the mechanical behaviour of those regions in reinforced concrete structures with static and geometric discontinuities. In this case, strut-and-tie models were derived from the principal stress trajectories of linear elastic stress fields, generated through finite element analysis (Schlaich et al. [18]). More recently, several computational frameworks have been introduced to automatise the generation of these models, mostly based on topology optimisation of ground trusses (Ali and White [1]; Liang et al. [10]) or finite element analysis (Biondini et al. [2]; Muttoni et al. [15]).

On the other hand, stress fields sprang out of the theory of plasticity (Prager and Hodge [17]; Gvozdev [6]; Drucker [5]) and applied to the analysis and design of reinforced concrete (Muttoni et al. [16]). Stress fields can be used for describing the stress distribution in the compression concrete struts and define the required layout of the steel reinforcement. Moreover, they are particularly suitable for the dimensioning of concrete members, detailing of discontinuity regions, and designing of the nodal geometries. Approaches for the automatic generation of 2D discrete and continuous stress fields have been proposed in the last few years (Hajdin [7]; Kostic [9]; Muttoni et al. [15]; Mata-Falcón et al. [11]). Although a clear connection between 2D discrete stress fields and graphic statics has been already pointed out (Muttoni et al. [16], Zanni and Pennock [20]), no direct approaches that can be also generalised to 3D, have been developed yet.

1.3 Yield lines

Yield line theory has been developed for the analysis of reinforced concrete slabs to identify their maximum load capacity. The theory follows the upper bound theorem of the theory of plasticity and requires the postulation of compatible mechanisms that generate rigid regions intersecting at yield lines. Assessing whether a proposed mechanism is compatible or not (Moy [14]) may prove a difficult task. A systematic method to deal with this problem has been proposed by Denton [4], and previously described, in more general terms, by Calladine [3] as part of the 'static-geometric analogies.' This method is based on the truss analogy, namely on the analogy between a compatible yield line mechanism and a self-stressed truss, the members of which lie on the mechanism's hinge lines. Advantages of this approach include its applicability in cases of concrete slabs with free edges and internal supports. Thus, a compatible mechanism of yield lines satisfies both requirements: rotational compatibility of the yield line mechanism and equilibrium of a corresponding self-stress truss. As a result, the compatibility of a yield line mechanism can be checked by enquiring the geometry of its corresponding truss. If the latter is self-stressed (without any external loads) and in static equilibrium, then the mechanism is compatible. Since the number of states of self-stress is equal to the number of degrees of freedom (DoFs) of the corresponding mechanism [4], from the upper bound theorem it follows that the yield line mechanisms of concrete slabs that are of interest have one DoF [4]. In addition, it suffices for the yield line mechanism to be a part or subset of a bigger truss for it to be compatible. As a result, the members of the equivalent truss can extend beyond the boundaries of the slab. Recent research has linked the truss analogy to graphic statics through the Airy stress function. As pointed out by Williams and McRobie [19], since a self-stress 2D truss is a projection of a polyhedral Airy stress function, a yield line mechanism is compatible if it is a projection of a polyhedron.

1.4 Objectives and contributions

In this paper, we discuss 2D and 3D stress fields in reinforced concrete from a geometrical point of view, based on the theory of plasticity. By using graphic statics and polar transformations (Konstantatou et al. [8]), we automatise the generation of 2D stress fields for a given boundary of material and an initial input strut-and-tie topology, and we extend these constructions to 3D. In the same context, we study 2D yield lines and give insights into their geometrical admissibility and the graphical representations of internal and external work. Moreover, we highlight how both stress fields, which are based on the lower bound theorem, and yield lines, which are based on the upper bound theorem of the theory of plasticity, are underpinned by the same fundamental geometric constructions.

The main tools used in this work are all four reciprocal objects of graphic statics, namely reciprocal form and force diagrams, their corresponding stress functions [8], and the combination thereof in terms of Minkowski sums [20], [12]. This approach allows a direct, interactive, and purely geometrical framework to be developed, that works in a unified way both for 2D and 3D compression-and-tension cases, while allowing for analysis and design freedom. Since this direct method of static equilibrium is based on projective geometry, it does not need any iteration for the construction of the reciprocal diagrams, and the process can be initialised from any of the four reciprocal objects. Moreover, the proposed approach works for any geometry of 2D trusses in static equilibrium and for polyhedral 3D trusses. The force diagrams for the 2D case are Maxwell 2D reciprocals (form edges correspond to perpendicular force edges, form nodes to force polygons) and for the 3D case are Rankine 3D reciprocals (form edges correspond to perpendicular force faces, form nodes to force polyhedra).

2 Direct graphical generation of discrete stress fields

The proposed method for the generation of discrete stress fields within a given boundary of material and an initial input strut-and-tie topology is based on the following steps:

- We initially set the input: topology of the strut-and-tie network; geometric boundary constraints; magnitude, direction, and points of application of the external forces (Fig. 1.a).
- We choose an initial form diagram ***P****(v, e, f)*, which has the same topology as the stress field/ Minkowski sum – i.e. every node of ***P*** is mapped to a force polygon in the stress field/ reciprocal force diagram ***P'****(v', e', f')*. The initial form diagram should not necessarily be in static equilibrium.
- We replace the external forces in the geometry of the initial form diagram with an auxiliary structure to produce an equivalent self-stressed truss (Fig. 1.b).
- We lift this self-stressed truss to the three-dimensional space [8] to produce the corresponding polyhedral Airy stress function (Fig. 1.d). This lifting process of the form diagram ensures that the corresponding strut-and-tie model is in static equilibrium by adjusting any nodes of the initially chosen geometry. Thus, the designer only needs to choose a topology and define an equivalent self-stressed truss, which then will be automatically corrected to a truss in static equilibrium. Based on the geometry and curvature of the polyhedral Airy stress function (valley folds or ridges), the struts in compression, and the ones in tension are clearly identified.
- Through polar transformations, we then map the polyhedral stress function to its reciprocal polyhedron (Fig. 1.e), which we project subsequently one dimension down to obtain the reciprocal force diagram in a Maxwell 2D configuration (Fig. 1.c).
- Reciprocal form and force diagrams are combined into a Minkowski sum, which is topologically the same as the stress field (Fig. 2.a). This Minkowski sum is now a stress field for uniform hydrostatic stress-state under the given external forces (Fig. 2.a). However, it corresponds to a case where the application points of the external forces (but not their magnitude) may differ in comparison to the given input (Fig. 2.b).
- In order to fit the stress field within the material boundaries and to the specific points of application of the external forces, we apply geometric transformations to the polyhedral Airy stress function of the form diagram until the Minkowski sum conforms to the given geometrical requirements (Fig. 2.d). In the general case, the way to relocate the force polygons of the stress field to the desired locations, without altering the direction of the lines of application and magnitude of external forces, is by parallel translations of the faces of the form Airy stress function. As a result, the magnitudes of the external forces remain the same (the dihedral angles between the faces of the form Airy stress function do not change) but their point of application changes in the stress field. In the simple but common 2D case where the applied loads are vertical, these operations are global affine transformations such as 1D scaling.
- It should be noted that the resulting strut-and-tie network has more vertices than the initial 'form' diagram. That is, some of the nodes of the initial form diagram are actually clusters of overlapping nodes that become distinct nodes in the new strut-and-tie network. This effect is due to a subtle difference between the Minkowski sum and the stress field. That is, the initial Minkowski sum represents the static equilibrium of a given form diagram for which every n-sided force polygon corresponds to a node with n number of concurrent form edges (truss analogy). However, these form edges do not necessarily pass from the midpoint of the force edges – they are just perpendicular to them. As a result, the initial form edges cannot be seen as the lines of action of the resultants. The force polygon is still in hydrostatic static equilibrium, but the resulting resultants are different.

In the general case, the stress fields in reinforced concrete can be diffused or non-perpendicular to the nodes (Fig. 2.e,f), as long as the stress resultants are kept constant in terms of location and magnitude. Consequently, along with the force diagram (Fig. 2.c), which corresponds to the static equilibrium of the resultants, we have the flexibility to define the *stress diagram*, which depicts the particular stress distribution given a static equilibrium of force resultants (Fig. 2.e). If we assume hydrostatic stress state for all the nodes of the stress field, then the presented method readily works for compression-only cases (CCC nodes) where the compressive struts do not intersect each other. However, the corresponding generic force diagram needs to be changed in other cases where compressive struts meet tensile reinforcement (CCT and CTT nodes) (Fig. 2.b). As a result, by rearranging the order of the edges of the force polygon, we can ensure appropriate anchorage of the tensile reinforcement for CCT and CTT nodes [16].

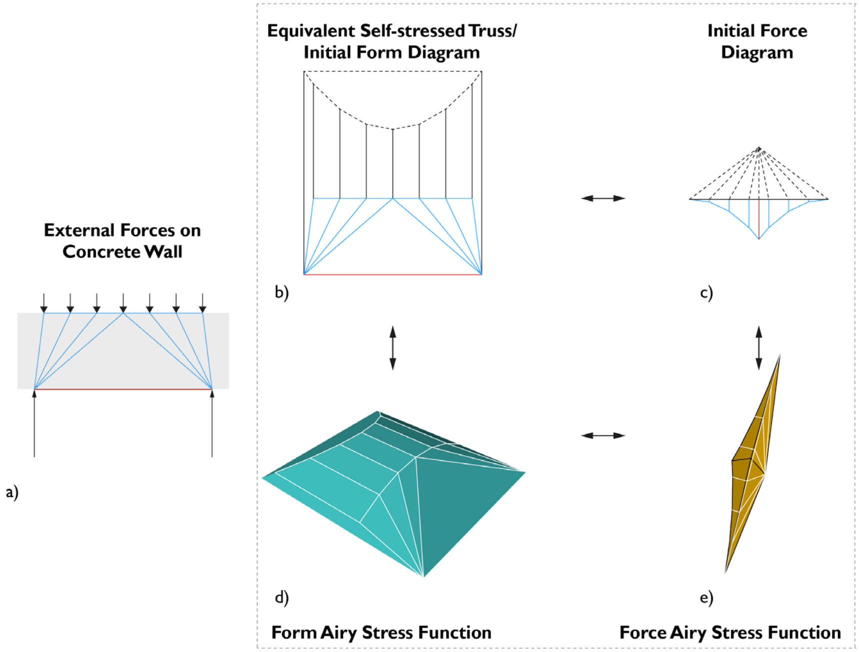

Fig. 1 a) External forces and reactions on a concrete wall; b) Equivalent geometry of a self-stressed truss including a funicular element deriving from the force polygon (initial form diagram); c) Reciprocal (initial force diagram); d) Corresponding polyhedral Airy stress function of (b); e) Corresponding polyhedral Airy stress function of (c).

3 Direct graphical generation of yield line patterns

Accordingly, we analyse and design yield line mechanisms by using the polyhedral Airy stress function of the corresponding truss and the resulting Minkowski sum. This geometrical method can be used to check for compatibility of the mechanism while providing an intuitive and straightforward representation of the internal and external work. The proposed method is as follows:

- We initially set the input: 2D geometry of the hinge lines of the proposed mechanism, which gives the underlying truss geometry ***P****(v, e, f)*.
- We perform a check to assess if a plane-faced polyhedron is generated when ***P*** is lifted one dimension up to create the corresponding 3D Airy stress function [8]. If it is the case, ***P*** corresponds to a compatible yield line mechanism. If it is not the case, we impose the planarity of the faces of the Airy stress function; accordingly, we adjust the coordinates of the nodes of ***P*** to generate ***P'***, which represents a compatible mechanism.
- We define the angles between adjacent faces of the polyhedral Airy stress function to be equal to the rotational angles between adjacent rigid regions of the mechanism. We reciprocate the polyhedron using a polar transformation, and after obtaining the reciprocal of ***P*** (respectively ***P'***), we construct the corresponding Minkowski sum. The surface area of each element of the Minkowski sum is equivalent to the length of the corresponding yield line multiplied by the rotational angle

of that line. Hence, the Minkowski sum is the geometrical interpretation of the internal work W_I, which is expressed by Equation 1 [14]. W_I is the sum, over all the n yield lines, of the rotation angle θ_i around each yield line multiplied by the moment M_i along that line.

- The external work W_E is expressed by Equation 2 [14]. W_E is the sum, over all the rigid regions, of the integral of the load q_i applied on each element multiplied by its corresponding displacement Δ_i. Thus, W_E is the volume of the polyhedral Airy stress function multiplied by the scalar q_i. The equivalence between the external work over the rigid regions and the internal work over every yield line is represented by Equation 3. Hence, the volume of the Airy stress function is equivalent to the surface area of the Minkowski sum.

$$W_I = \sum\left[\theta_i \int_S M_i ds\right] \qquad W_E = \sum\left[\int_A q_i \Delta_i dA\right] \tag{1) (2}$$

$$\sum\left[\int_A q_i \Delta_i dA\right] = \sum\left[\theta \int_S M_i ds\right] \tag{3}$$

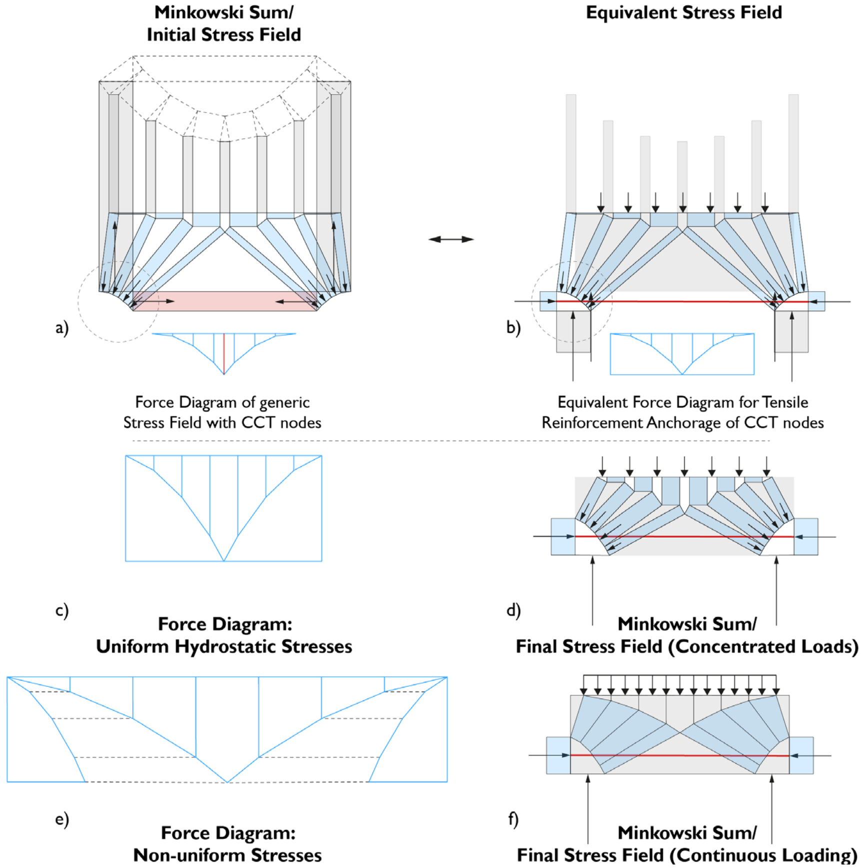

Fig. 2 a) Minkowski sum (initial generic stress field); b) Transformation of the nodal geometry of the CCT nodes to accommodate tensile reinforcement anchorage. Superposition of this stress field with the boundaries of the wall shows that this is not an acceptable solution; d) The final stress field coincides with the material boundary following transformations of (1.d); c) Corresponding force diagram for uniform hydrostatic stresses when the loads are concentrated (1.a); f,e) Transformation of the stress field between concentrated and continuous loading and corresponding changes in the force diagram for non-uniform stresses.

4 Further applications and extensions

In the example of Fig. 3, a discrete stress field is generated using the proposed method based on graphic statics and polar transformations (Section 2) for a given strut-and-tie topology, geometric boundary, and points of application of the external forces (Fig. 3.a). Through simple geometric transformations of the polyhedral Airy stress function (Fig. 3.f) a valid stress field can be found (Fig. 3.e) while automatically guaranteeing global static equilibrium throughout the process. The proposed method can be easily extended to obtain valid stress fields in 3D. In the 3D example of Fig. 4, the equivalent truss is lifted one dimension up to a plane-faced 4-polytopic stress function (Fig. 4.e) which in turn directly yields a Rankine 3D reciprocal (Fig. 4.c). Following the same method as in the 2D case (Section 2), the Minkowski sum provides a uniform hydrostatic stress field within the boundaries of the concrete block (Fig. 4.d). Eventually, we implement the above method to generate yield patterns (Section 3) to common cases that are found in the literature (Fig. 5). For each case, the equivalent truss geometry is first found. Then the polyhedral Airy stress function is used to check for compatibility of the mechanism while giving a visual representation of the external work. Through a polar transformation, the reciprocal stress function and the Minkowski sum can be generated, giving a visual representation of the internal work.

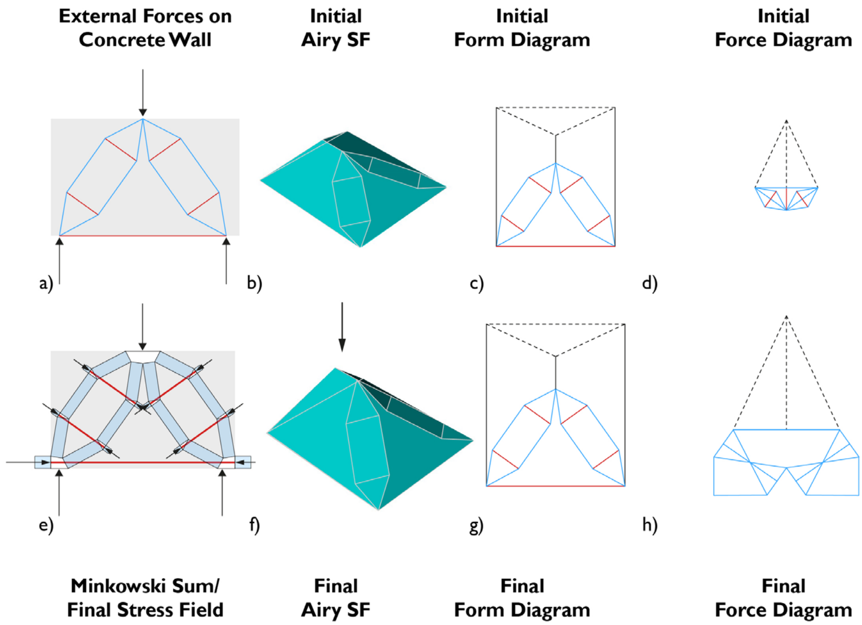

Fig. 3 A common stress field case is solved via the proposed method. Simple geometrical manipulations of the Airy stress function and changes in the resulting force diagram give a valid stress field with compression-only nodes.

5 Conclusions

The methods presented in this paper highlight how graphic statics is a valuable tool to solve important engineering problems related to reinforced concrete. In particular, the paper showed how graphic statics could be used for the direct and automatic generation of discrete 2D and 3D stress fields and for the geometrical representation of internal/external work and definition of compatible yield line mechanisms. Although stress fields in reinforced concrete are underpinned by the lower bound theorem of the theory of plasticity, whereas yield lines by the upper bound theorem, the same fundamental graphical approach can be used to generate both. In spite of their simplicity, the presented methods are extremely powerful and advantageous over other approaches such as the finite element analysis, in the sense that are visual, intuitive, and can be easily used for both analysis and design.

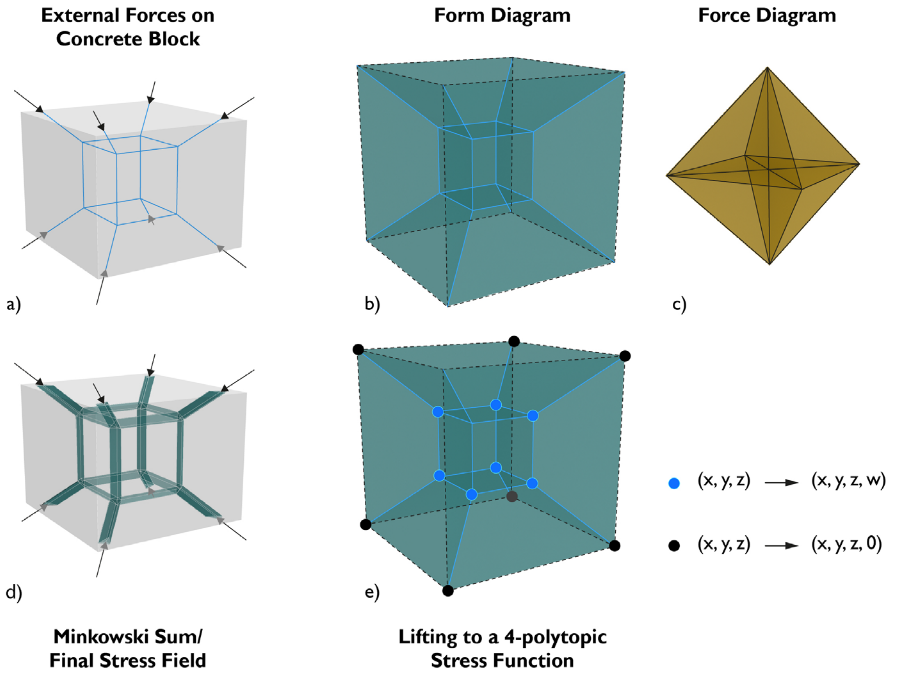

Fig. 4 a) A common strut-and-tie case for a cubic concrete block; b) Corresponding form diagram as a polyhedral spatial truss; c) Reciprocal Rankine 3D force diagram; d) Resulting discrete 3D stress field derived from a spatial Minkowski sum; e) Polyhedral form diagram, transformed into the 4-polytopic stress function by lifting its internal nodes to 4D.

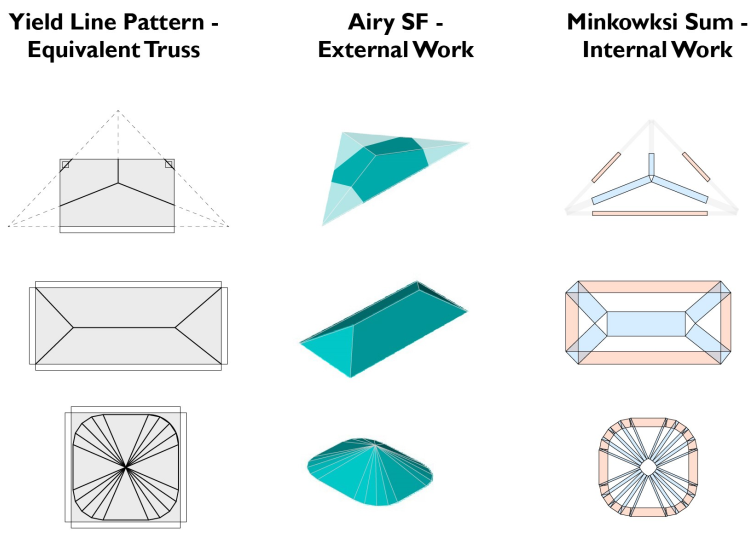

Fig. 5 left) Common cases of yield line patterns; centre) Corresponding polyhedral Airy stress function used for checking the compatibility of the mechanisms while providing visual interpretation of the external work; right) Corresponding Minkowski sums giving intuitive representation of the internal work and of compression-tension areas.

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